HOSSAM GHANEM

(33) 4.2 The Rolle's Theorem (B)

Example 7 19 July 29, 2000	Show that equation $x^7 + 5x + 3 = 0$ has exactly one real root

Solution

$let \ \ f(x) = x^7 + 5x + 3$
$\therefore \ f$ cont. on R
f diff. on R
$f(0) = 3 > 0$
$f(-1) = -1 - 5 + 3 = -3 < 0$
$\therefore \ \exists \, c \in (-1,0)$ such that $f(c) = 0$ $\qquad$ *I.V.T*
$\therefore$ the equation $f(x) = 0$ has a real root in $(-1, 0)$
$f^{\backslash}(x) = 7x^6 + 5 > 0$

let c_1 & c_2 are two real roots of the equation $f(x) = 0$ such that $c_1 \leq c_2$
f cont. on $[c_1, c_2]$
f is diff. on (c_1, c_2)
$f(c_1) = f(c_2) = 0$
$\therefore \ \exists \ K$ *such that* $f^{\backslash}(K) = 0$ $\qquad$ Rolle's Theorem
but $f^{\backslash}(x) \neq 0$
$\therefore \ c_1 = c_2$
$\therefore$ the equation $f(x) = 0$ has exactly one real root

Example 8 5 July 13, 1992	Let $f(x) = \tan x + 4x - 3$. Use the Rolle's Theorem to prove that the equation $f(x) = 0$ has only one real root in $\left(0, \frac{\pi}{4}\right)$

Solution

$f(x) = \tan x + 4x - 3$
$\therefore \ f$ cont. on $\left[0, \frac{\pi}{4}\right]$
$f(0) = 0 + 0 - 3 = -3 < 0$
$f\left(\frac{\pi}{4}\right) = 1 + \pi - 3 > 0$
$\therefore \ \exists \, c \in \left(0, \frac{\pi}{4}\right)$ such that $f(c) = 0$ $\qquad$ I.V.T
$\therefore$ The equation $f(x) = 0$ has a real solution in $\left(0, \frac{\pi}{4}\right)$
$f^{\backslash}(x) = \sec^2 x + 4 > 0$
Let c_1 & c_2 are real solutions in $\left(0, \frac{\pi}{4}\right)$ such that $c_1 \leq c_2$
f cont. on $[c_1, c_2]$
f diff. on (c_1, c_2)
$f(c_1) = f(c_2) = 0$
$\therefore \ \exists \ K \in (c_1, c_2)$ such that $f^{\backslash}(K) = 0$ $\qquad$ Rolle's Theorem
but $f^{\backslash}(x) > \ \forall \, x \in R$
$\therefore \ c_1 = c_2$
$\therefore \ f(x) = 0$ has only one solution

Example 9 33 May 6, 2004	Let f , $f^{\backslash}$ and $f^{\backslash\backslash}$ be continues functions on $[\,a\,,b\,]$, Suppose that the graph of f intersect the x – axis at 3 points in $(\,a\,,b\,)$. Show that $f^{\backslash\backslash}(x) = 0$ has a solution in $(\,a\,,b\,)$

Solution

$\because f^{\backslash\backslash}$ cont. on $[\,a\,,b\,]$

$\therefore$ f & $f^{\backslash}$ are diff. on $[\,a\,,b\,]$

let the graph intersect $x - axis$ at $c_1, c_2, c_3 \in (\,a\,,b\,)$ such that $c_1 < c_2 < c_3$

$\therefore f(c_1) = f(c_2) = f(c_3) = 0$

(i) f cont. on $[\,c_1\,,c_2\,]$

f diff. on $(\,c_1\,,c_2\,)$

$f(c_1) = f(c_2) = 0$

$\therefore$ $\exists\, K_1 \in (\,c_1\,,c_2\,)$ such that $f^{\backslash}(K_1) = 0$ Rolle's Theorem

(ii) f cont. on $[\,c_2, c_3\,]$

f diff. on $(\,c_2, c_3\,)$

$f(c_2) = f(c_3) = 0$

$\therefore$ $\exists\, K_2 \in (\,c_2, c_3\,)$ such that $f^{\backslash}(K_2) = 0$ Rolle's Theorem

$f^{\backslash}$ cont. on (K_1, K_2)

$f^{\backslash}$ diff. on (K_1, K_2)

$f^{\backslash}(K_1) = f^{\backslash}(K_2) = 0$

$\therefore$ $\exists\; m \in (K_1, K_2)$ such that $f^{\backslash\backslash}(m) = 0$ Rolle's Theorem

$\therefore$ $f^{\backslash\backslash}(x) = 0$ has a solution in $(\,a\,,b\,)$

Example 10 37 May 4, 2006	Use Rolle's Theorem to show that the graph of $f(x) = x^4 + x^3 - 3x^2 + x + 1$ Cannot have more than one inflection point in [0 , 1]

Solution

$f(x) = x^4 + x^3 - 3x^2 + x + 1$

$f^{\backslash}(x) = 4x^3 + 3x^2 - 6x$

$f^{\backslash\backslash}(x) = 12x^2 + 6x - 6$

$f^{\backslash\backslash\backslash}(x) = 24x + 9$

let f has two inflection points on (0,1) at $x = c_1$ and $x = c_2$ such that $c_1 \leq c_2$

$\therefore$ $f^{\backslash\backslash}(c_1) = f^{\backslash\backslash}(c_2) = 0$

$f^{\backslash\backslash}$ cont. on $[\,c_1\,,c_2\,]$

$f^{\backslash\backslash}$ diff. on $(\,c_1\,,c_2\,)$

$f^{\backslash\backslash}(c_1) = f^{\backslash\backslash}(c_2)$

$\therefore \exists\, K \in (0,1)$ *such that* $f^{\backslash\backslash\backslash}(K) = 0$ Rolle's Theorem

but $f^{\backslash\backslash\backslash}(x) = 24x + 6 \;\; > 0 \;\; \forall\, x \in [0,1]$

$\therefore$ $c_1 = c_2$

$\therefore$ f Cannot have more than one inflection point in [0 , 1]

Example 11 46 August 1 , 2009	Use Rolle's Theorem to show that for all $a \in \mathbb{R}$ the equation $$\frac{3x}{2} + \sin x = a$$ has at most one solution

Solution

Let $f(x) = \frac{3x}{2} + \sin x - a$

$f^\backslash(x) = \frac{3}{2} + \cos x \quad > \quad 0$

f cont. on R

f diff. on R

let c_1 & c_2 are real roots for the equation $f(x) = 0$ such that $c_1 < c_2$

f cont. on $[\, c_1 , c_2 \,]$

f diff. on $(\, c_1 , c_2 \,)$

$f(c_1) = f(c_2) = 0$

$\therefore \ \exists\ K \in (\, c_1 , c_2 \,)$ such that $f^\backslash(K) = 0$ Rolle's Theorem

but $f^\backslash(x) > 0 \ \ \forall\ x \in R$

$\therefore \ f(x) = 0$ has at most one solution

$\therefore \ \frac{3x}{2} + \sin x - a \ = 0$ has at most one solution

$\therefore \ \frac{3x}{2} + \sin x = a$ has at most one solution

Example 12 47 December 22, 2009	Let $f(x) = x^2 - x + \cos x$. Show that the equation $f'(x) = 0$ has exactly one solution

Solution

$f(x) = x^2 - x + \cos x$

$f^\backslash(x) = 2x - 1 - \sin x$

$f^{\backslash\backslash}(x) = 2 - \cos x > 0$

$f^\backslash(0) = -1 < 0$

$f^\backslash(\pi) = 2\pi - 1 - 0 > 0$

$f^\backslash$ cont. on $(0 , \pi)$

$\therefore \ \exists\ c \in (0, \pi)$ such that $f^\backslash(c) = 0$ I.V.T

$\therefore$ the equation $f^\backslash(x) = 0$ has a real root in $(0, \pi)$

Let c_1 , c_2 are real roots in $(0, \pi)$ for $f^\backslash(x) = 0$ such that $c_1 \leq c_2$

$f^\backslash$ cont. on (c_1 , c_2)

$f^\backslash$ diff. on (c_1 , c_2)

$f^\backslash(c_1) = f^\backslash(c_2) = 0$

$\exists\ K \in (\, c_1 , c_2 \,)$ *such that* $f^{\backslash\backslash}(K) = 0$ Rolle's Theorem

but $f^{\backslash\backslash}(x) \ > 0 \ \ \forall\ x \in R$

$c_1 = c_2$

$\therefore$ the equation $f^\backslash(x) = 0$ has exactly one real solution

Homework

1	14 December 18, 1997 Show that equation $x^3 + 3x + 1 = 0$ has exactly one real root
2	38 January 15, 2011 Show that equation $f(x) = x^5 + x^3 - 1$ exactly one real root in $[\,0\,,1\,]$.
3	16 July 30, 1998 Show that equation $x^3 + x - 17 = 0$ can not have two distinct real roots
4	Show that equation $x^3 + 5x + c = 0$ has exactly one real root
5	Show that equation $4x^3 - 6x^2 + 4x - 1 = 0$ has exactly one real root in $(0\ ,\ 1)$
6	34 July 22, 2004 Show that $f(x) = x^4 + 2x^2 - 3x + 1$ has exactly one critical number
7	31 June 5, 2008 Show that equation $2x^5 + x - 1 = 0$ has exactly one real solution
8	If $f(x) = x^5 + 9x + 1$, Show that the equation $f(x) = 0$ has exactly one real root.
9	Let $f(x) = x^5 + 9x + 1$. Show that $f(x)$ is increasing function and the equation $f(x) = 0$ has exactly one real root
10	Show that equation $x^5 + ax + 3 = 0$, $a > 0$ has exactly one real root
11	44 December 21, 2008 Use Rolle's Theorem to show that the equation $x^7 + 4x^3 + x - 2 = 0$ cannot have two different roots.
12	49 July 24, 2010 (3 Points) Use Rolle's Theorem to show that the equation $4x^3 + 4x + 3 = 0$ Cannot have more than one real root
13	Show that equation $x^5 + 5x + 10 = 0$ has exactly one real root

Homework

14	25 January 12 .2003 Let $f(x) = x\cos x$. Use Rolle's theorem to show that $f^{\backslash\backslash}(c) = 0$ for some $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
15	41 7 January 2012 [4 Pts.] Show that the equation $x^3 - 3x^2 + 5x = 7$ has exactly one real root .

13	Show that equation $x^5 + 5x + 10 = 0$ has exactly one real root

Solution

Let $f(x) = x^5 + 5x + 10$

$f^{\backslash}(x) = 5x^4 + 5 \quad > 0$

$\therefore$ f cont. on R

f diff. on R

$f(0) = 10 \quad > 0$

$f(-2) = (-2)^5 + 5(-2) + 10 = -32 \quad < 0$

$\therefore \exists\, c \in (-2,0)$ such that $f(x) = 0$ I.V.T

Let c_1 & c_2 are two real roots such that $c_1 \leq c_2$

$f(c_1) = f(c_2) = 0$

f cont. on $[c_1, c_2]$

f diff. on (c_1, c_2)

$\therefore \exists\ K$ *such that* $f^{\backslash}(K) = 0$ Rolle's theorem

but $f^{\backslash}(x) \neq 0$

$\therefore c_1 = c_2$

$\therefore$ the equation $f(x) = 0$ has exactly one real root